ON ASYMPTOTIC STABILITY AND INSTABILITY OF MOTION WITH RESPECT TO A PART OF THE VARIABLES

PMM Vol. 35, №1, 1971, pp. 138-143
V. V. RUMIANTSEV
(Moscow)
(Received July 7, 1970)

Study of the stability of motion with respect to a part of the variables [1] finds application in various problems, particularly in those of the motion of systems with cyclic coordinates, of nonholonomic systems, and others. The method of Liapunov functions has also been found to be effective in the present problem [3–5].

Below we prove several theorems on the asymptotic stability and instability with respect to a part of the variables, representing generalizations of certain known theorems on the Liapunov method. Two examples are given.

Let us consider the following equations of perturbed motion of a system:

$$dx_s / dt = X_s(t, x_1, \ldots, x_n) \qquad (s = 1, \ldots, n) \tag{1}$$

where x_s are real variables characterizing the deviations of the system from the unperturbed motion, t is time and $X_s(t, x_1, \ldots, x_n)$ are real functions defined and continuous for all $t \geqslant 0$ in some region G containing the point $x = 0$, of the space $\{x_s\}$. We assume that the functions $X_s(t, x_1, \ldots, x_n)$ satisfy in G the conditions of existence and uniqueness of the solutions $x_s(t; t_0; x_{10}, \ldots, x_{n0})$ of (1). These solutions depend continuously on $t_0 \geqslant 0$ and x_{r0} $(r = 1, \ldots, n)$ and become equal to x_{s0} at $t = t_0$, moreover $X_s(t, 0, \ldots, 0) \equiv 0$.

Let us consider the stability of the unperturbed motion $x = 0$ with respect to a part of the variables [1], choosing for definiteness the variables $x_1, \ldots, x_k$ $(0 < k \leqslant n)$. We shall denote these variables by $y_i = x_i$ $(i = 1, \ldots, k)$, and the remaining $m = n - k \geqslant 0$ variables by $z_j = x_{k+j}$ $(j = 1, \ldots, m)$, i.e. we shall write the vector x in the form $x = (y_1, \ldots, y_k, z_1, \ldots, z_m)$ and call the stability with respect to $x_i = y_i$ $(i = 1, \ldots, k)$ the y-stability.

We consider the region G of the form

$$\| y \| = \Big(\sum_{i=1}^{k} y_i^2\Big)^{1/2} < H = \text{const}, \qquad \| z \| = \Big(\sum_{j=1}^{m} z_j^2\Big)^{1/2} < \infty \tag{2}$$

When studying the asymptotic stability with respect to a part of the variables, we shall follow the example of [2] in including the estimate of the region of attraction G_λ in the definition of the asymptotic stability. The motion $x = 0$ shall be called asymptotically y-stable and the region G_λ of the space $\{x_s\}$ lies in the region of y-attraction of the point $x = 0$, if the motion is y-stable and the following conditions hold:

$$\lim y(t; t_0, x_0) = 0, \quad y(t; t_0, x_0) \in \Gamma_y \text{ when } t \geqslant t_0, \; x_0 \in G_\lambda \tag{3}$$

Here Γ_y is some region of G specified in advance and G_λ is the region of perturbations. We shall define for simplicity the regions Γ_y and G_λ by the conditions

$$\| y \| \leqslant A < H \text{ and } \| x_0 \| = \Big(\sum_{s=1}^{n} x_{s0}^2\Big)^{1/2} < \lambda, \;\; \lambda > 0$$

respectively, and consider the region $\Gamma \supset \Gamma_y$ defined by the inequalities

$$\| y \| \leqslant A, \qquad \| z \| < \infty \tag{4}$$

When $H = \infty$, the motion $x = 0$ is asymptotically y-stable in the large, provided that it is y-stable and the condition $\lim y(t, t_0, x_0) = 0$ for $t \to \infty$ holds for any $\| x_0 \|$ no matter how large.

We consider the functions $V(t, x)$ defined and continuous together with their first order partial derivatives in the region (4) for all $t \geqslant 0$ satisfying the condition $V(t, 0) = 0$ for all $t \geqslant 0$, and their derivatives $V^{\cdot}(t, x)$ with respect to time arising from the equations of perturbed motion.

We recall that $V(t, x)$ is y-positive-definite in Γ, if a positive-definite function $w(y)$ exists such that the following inequality holds:

$$V(t, x) \geqslant w(y) \quad \text{for} \quad x \in \Gamma, \ t \geqslant 0 \tag{5}$$

The function $V(t, x)$ admits [2] an upper bound in y within the region Γ, the bound becoming infinitely small at $x = 0$ (or more concisely, it has an infinitely small upper bound in y) if a continuous function $W(y)$ exists satisfying the conditions

$$| V(t, x) | \leqslant W(y) \quad \text{for} \quad x \in \Gamma, \ t \geqslant 0, \ W(0) = 0 \tag{6}$$

The sufficient conditions of y-stability are furnished by the theorem on stability with respect to a part of the variables [3] which admits a converse resembling the Liapunov theorem on stability.

We now consider the problem of asymptotic y-stability, introducing the notation $V(t) = V(t, x(t; t_0, x_0))$.

Theorem 1. If Eqs. (1) are such that a y-fixed-sign function $V(t, x)$ can be found in the region Γ, whose derivative $V^{\cdot}(t, x)$ with respect to time is (by virtue of these equations) a constant-sign function in the region Γ having the sign opposite to that of V, and furthermore $V(t) \to 0$ as $t \to \infty$ and if the inequality

$$\sup [V(t_0, x); \ \| x \| \leqslant \lambda] < \inf [V(t, x); \ \| y \| = A_1 < A \quad \| z \| < \infty, \ t_0 \leqslant t < \infty] \tag{7}$$

holds, then the motion $x = 0$ is asymptotically y-stable and the region G_λ lies in the region of y-attraction of the point $x = 0$.

Proof. Let $V(t, x)$ be a y-positive-definite function. Then a positive-definite function $w(y)$ can be found such, that the inequality (5) holds in Γ and $V^{\cdot}(t, x) \leqslant 0$. Let us set

$$l = \inf [V(t, x); \ \| y \| = A_1 \ \| z \| < \infty, \ t_0 \leqslant t < \infty]$$

Since $V(t_0, x)$ is independent of t, it admits an infinitely small upper bound and it follows that such λ can be found for l, that $V(t_0, x) < l$ when $\| x \| \leqslant \lambda$. Imposing the condition that $\| x_0 \| < \lambda$ on the initial values x_0 we obtain, by virtue of the properties of $V(t, x)$, the following inequalities:

$$w(y) \leqslant V(t, x) \leqslant V(t_0, x_0) < l$$

from which it follows that $\| y \| < A$ for all $t \geqslant t_0$. Consequently, if for given A_1 and λ $(A_1 > \lambda)$ the condition that $V(t) \to 0$ as $t \to \infty$ and condition (7)both hold, then $w(y(t, t_0; x_0)) \to 0$ as $t \to \infty$ and this, in turn, implies that $\| y(t; t_0; x_0) \| \to 0$ as $t \to \infty$, Q. E. D.

Theorem 1 represents a generalization of a number of known theorems on asymptotic stability with respect to a part of the variables [3–5] whose conditions ensure that

$V(t) \to 0$ as $t \to \infty$. The formulation of these theorems may also include the estimates of G_λ similar to (7). Thus e.g. we have

Theorem 2. If Eqs. (1) are such that a y-fixed-sign function $V(t, x)$ admitting an infinitely small upper bound y (in x) exists in the region Γ, and its derivative $V^{\cdot}(t, x)$ with respect to time is a fixed-sign function in y (in x) in the region Γ, and has the sign opposite to that of V, furthermore if condition (7) is satisfied, then the motion $x = 0$ us asymptotically y-stable and region G_λ lies in the region of y-attraction of the point $x = 0$.

Theorems of Krasovskii [2] and Chetaev [6] on asymptotic stability can also be extended to embrace asymptotic stability with respect to a part of the variables.

Theorem 3. If Eqs. (1) are such that their right-hand sides $X(t, x)$ are ϑ periodic functions of time t or do not depend explicitly on t, the solutions $z_j(t; t_0, x_0)$ are bounded for all $\|x_0\| \leqslant \lambda$ and a y-positive-definite function $V(t, x)$ exists in Γ, which is ϑ periodic in t or does not depend explicitly on t, satisfying the inequality (7), and furthermore, if its time derivative satisfies the conditions that $V^{\cdot}(t, x) \leqslant 0$ in Γ and $V^{\cdot}(t, x) = 0$ only at the points of the set M, the latter containing only parts of the trajectories of the system (1) with exception of the solution $x = 0$, then the motion $x = 0$ is asymptotically y-stable and G_λ lies in the region of y-attraction of the point $x = 0$.

Proof. Function $V(t, x)$ satisfies the conditions of the theorem on stability with respect to a part of the variables, therefore $\|y\| < A$ when $t \geqslant t_0$ for all $\|x_0\| \leqslant \lambda$. The solutions $z_j(t; t_0, x_0)$ are bounded by the condition of the theorem, consequently all solutions $x_s(t; t_0\, x_0)$ of (1) are also bounded for all $\|x_0\| \leqslant \lambda$. Function $V(t)$ is a monotonous, nonincreasing function of time, consequently $\lim V(t) = V^*$ when $t \to \infty$ exists and $V(t) \geqslant V^*$ for all $t \geqslant t_0$. Let us consider the sequence of points $x^{(k)} = x(t_0 + \kappa\vartheta,\ t_0; x_0)$ $(k = 1, 2, \ldots)$, where ϑ is the period of $X(t, x)$ with respect to time or is any positive number if X does not depend explicitly on time. The bounded sequence of points $x^{(k)}$ has a limit point $x = x^*$ and the relation $V^* = V(t_0, x^*)$ holds by virtue of the continuity and periodicity of $V(t, x)$. From this stage the proof of the equation $V^* = 0$ follows that given in the Krasovskii theorem [2].

We note that according to Yoshidzawa [7] the sufficient condition for all solutions $x(t; t_0; x_0)$ (or solutions $z(t; t_0, x_0)$) of (1) to be bounded is, that a function $V(t, x)$ exists such that $V(t, x) \geqslant w(x)$ (or $V(t, x) \geqslant w(z)$) where $w(x) \to \infty$ as $\|x\| \to \infty$ ($w(z) \to \infty$ as $\|z\| \to \infty$) and $V^{\cdot}(t, x) \leqslant 0$.

Theorems 1 and 3 can be extended to embrace the asymptotic y-stability in the large, provided that $H = \infty$ in (2). Moreover the conditions of these theorems must be supplemented with the requirement that the functions $V(t, x)$ admit an infinitely small upper bound [2] in y as $\|y\| \to \infty$, i.e. that the positive-definite function $w(y)$ in condition (5) satisfies the condition that $\lim w(y) = \infty$ as $\|y\| \to \infty$. Then any region $w(y) \leqslant V_0$ will be bounded and the proof of the asymptotic y-stability follow that of Theorems 1—3. Of course, it is assumed here that $V(t, x)$ in Theorem 2 admits an upper limit in y (in x) over the whole space $\|x\| < \infty$, i.e. a continuous function $W(y)$ ($W(x)$) exists such that

$$V(t, x) \leqslant W(y) \quad (V(t, x) \leqslant W(x)), \quad W(0) = 0$$

and the solutions $z_j(t; t_0, x_0)$ in Theorem 3 are bounded for all $\|x_0\| < \infty$.

Theorem 4. If Eqs. (1) are such that a function $V(t, x)$ exists and satisfies the following conditions [6]:

(1) Function

$$V(t,\ x) - \theta(t)\, w(y) \qquad (\theta(t_0) = 1)$$

is always positive when $w(y)$ is positive-definite and independent of time and the function $\theta(t)$ tends monotonously to infinity with increasing t, and

(2) by virtue of these equations its derivative $V^{\cdot} \leqslant 0$, then the motion $x = 0$ is asymptotically y-stable and the range of possible values of y_i is defined by the inequality

$$w(y) \leqslant V_0 / \theta(t), \qquad V_0 = V(t_0,\ x_0) \tag{8}$$

Proof. By the conditions of the theorem we have the following inequalities:

$$\theta(t)\, w(y) \leqslant V(t,\ x) \leqslant V_0$$

from which (8) follows. Since $\theta(t) \to \infty$ as $t \to \infty$, then $w(y(t;\ t_0;\ x_0)) \to 0$ and $\| y(t;\ t_0;\ x_0) \| \to 0$ as $t \to \infty$.

We now assume that the right-hand sides of (1) are defined, continuous, bounded, and satisfy the conditions of existence and uniqueness of the solution in the region

$$t \geqslant 0, \quad \| x \| < H = \text{const} \tag{9}$$

Theorem 5. If Eqs. (1) are such that a fixed-sign function $V(t,\ x)$ exists in some region $\| x \| < H_1 < H$, whose derivative with respect to time $V^{\cdot}(t,\ x)$ is a y-fixed-sign function of sign opposite to that of V and if condition (7) holds, then the motion $x = 0$ is atable and asymptotically y-stable and G_λ lies in the region of y-attraction of the point $x = 0$.

The proof is elementary and is based on the Liapunov's theorem, and on the theorem given in [5].

Example 1. We consider the equations of motion of a mechanical system near its equilibrium position $q_i = q_i^{\cdot} = 0$ $(i = 1, \ldots, n)$ acted upon by potential gyroscopic and dissipative forces

$$\frac{d}{dt}\frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} = \frac{\partial U}{\partial q_i} + \sum_{j=1}^{n} g_{ij} q_j^{\cdot} + Q_i, \quad g_{ij} = -g_{ji} \quad (i, j = 1, \ldots n) \tag{10}$$

Total mechanical energy $H = T - U$ of the system does not depend explicitly on time. Let the dissipative forces $Q_i = Q_i(q_1, \ldots, q_n, q_1^{\cdot}, \ldots, q_n^{\cdot})$ satisfy the conditions

$$Q_i(q_i, \ldots, q_n;\ 0, \ldots, 0) \equiv 0, \quad Q_1 q_1^{\cdot} + \ldots + Q_n q_n^{\cdot} \leqslant 0$$

and let $Q_1 q_1^{\cdot} + \ldots + Q_n q_n^{\cdot} = 0$ if and only if all $q_i^{\cdot} = 0$ $(i = 1, \ldots, n)$.

From Eqs. (10) follows

$$dH/dt = Q_1 q_1^{\cdot} + \ldots + Q_n q_n^{\cdot}$$

this describes the dissipation of energy during any such motion of the system, for which $Q_1 q_1^{\cdot} + \ldots + Q_n q_n^{\cdot} \not\equiv 0$ when $t \geqslant t_0$. The energy of the system as nonincreasing function tends to some limit H^* where $H \geqslant H^*$ as $t \to \infty$.

If $H(q_1, \ldots, q_n, q_1^{\cdot}, \ldots, q_n^{\cdot})$ is a positive-definite function of q_i, $q_i^{\cdot}$ $(i = 1, \ldots, n)$, then at the beginning we have an isolated position of equilibrium asymptotically stable by Theorem 1 as the energy H dissipates until the system reaches the coordinate origin $(H^* = 0)$ [6]. If $H = H(q_1, \ldots, q_n, q_1^{\cdot}, \ldots, q_n^{\cdot})$ is a positive-definite function of its arguments $(k < n)$, then $H^* = 0$ and the initial position is asymptotically stable in and $q_i^{\cdot}$, q_s $(i = 1, \ldots, n;\ s = 1 \ldots, k)$.

If $H(q_1, \ldots q_n, q_1^{\cdot}, \ldots, q_n^{\cdot})$ is a positive-definite function of $q_i^{\cdot}$, q_s $(i = 1, \ldots, n;$

$s = 1, \ldots, k$) and some reasons (e. g. if $H \to \infty$ when $q_{k+1}^{\cdot 2} + q_{k+2}^{\cdot 2} + \ldots, q_n^{\cdot 2} \to \infty$) imply that the variables $q_{k+1}, \ldots, q_n$ remain bounded when $t \geqslant t_0$, then $H^* = 0$ and by Theorem 3 the initial position is asymptotically stable in $q_i^{\cdot}$ and q_s ($i = 1, \ldots, n$; $s = 1, \ldots, k$).

In certain cases the asymptotic stability with respect to a part of the variables can be deduced even when the boundedness with respect to the remaining variables is not known in advance. We consider a particular example [5] of the motion of a heavy mass point along a surface, with the viscous friction present, when the energy

$$H = {}^1/_2 (x^{\cdot 2} + y^{\cdot 2} + z^{\cdot 2}) + {}^1/_2\, g y^2 (1 + x^2)$$

is a positive-definite function of the variables $x^{\cdot}$, $y^{\cdot}$, y. By Theorem 1 the initial position $x=y=x^{\cdot}=y^{\cdot}=0$ is asymptotically stable in $x^{\cdot}$, $y^{\cdot}$, y variables.

Example 2. We consider a motion near the position of equilibrium $q_i = q_i^{\cdot} = 0$ ($i = 1, \ldots, n$) of a system with linear nonholonomic constraints, [8, 9] acted upon by potential and dissipative forces Q_i ($i = 1, \ldots, k$) of the same form as those in the previous example. Using the equations of motion in the form given by Voronets we obtain the following expression for the rate of dissipation of energy:

$$d/dt\,(T - U) = \sum_{i=1}^{k} Q_i q_i^{\cdot}$$

We assume that $Q_1 q_1^{\cdot} + \ldots + Q_k q_k^{\cdot} = 0$ when and only when all $q_i^{\cdot} = 0$ ($i = 1, \ldots, k$). If the force function U ($q_1, \ldots, q_n$) and the expression

$$\sum_{i=1}^{k} q_i \left(\frac{\partial U}{\partial q_i} + \sum_{r=k+1}^{n} \frac{\partial U}{\partial q_r} b_{ri} \right)$$

are both negative-definite with respect to the variables q_i ($i = 1, \ldots, k$), then the function $T - U$ is positive-definite with respect to q_i and $q_j^{\cdot}$ ($i = 1, \ldots, k$, $j = 1, \ldots, n$) and the generalized potential forces $Q_i^* \neq 0$ in the neighborhood of the position of equilibrium provided that $q_i \neq 0$ ($i = 1, \ldots, k$). Under these conditions dissipation of energy takes place until the time when all velocities $q_i^{\cdot}$ become equal to zero $q_i^{\cdot} = 0$ ($i = 1, \ldots, k$). It follows that if the variables q_r ($r = k + 1, \ldots, n$) remain bounded for $t \geqslant t_0$, then $T - U \to 0$ as $t \to \infty$ and by Theorem 3 the initial position is asymptotically stable in q_i and $q_i^{\cdot}$ ($i = 1, \ldots, k$).

We use the opportunity here to remark, that asymptotic stability of the position of equilibrium with respect to q_i and $q_i^{\cdot}$ ($i = 1, \ldots, k$) is established with the help of the function (3, 4) of [8] under the assumption that the fixed-sign property of U emerges from its quadratic term in the Maclaurin expansion.

Let us now consider the instability.

As we already noted in [3], the general theorem of Chetaev [6] on instability can be used in our study of the instability with respect to a part of the variables. It can easily be shown that this theorem allows the following formulation:

Theorem 6. If Eqs. (1) are such that a function V (t, y) can be found which is bounded in the region V (t, y) > 0 existing at any $t \geqslant t_0$ and for arbitrarily small absolute values of the variables y_i, and whose derivative $V^{\cdot}(t, x)$ is, by virtue of these equations y-positive-definite in the region $V(t, y) > 0$, then the motion $x = 0$ is

y-unstable.

The proof follows that given in [6] in every detail. As both Liapunov theorems on instability follow from the Chetaev theorem [6], so the analogs of the Liapunov theorem on y-instability follow from Theorem 6. On inspecting the function $V(t, y)$ it is easily seen that the analog of the first Liapunov theorem on instability is obtained from the Liapunov theorem under the condition that the derivative $V^{\cdot}(t, x)$ is a y-fixed-sign function. The analog of the second Liapunov theorem is formulated in exactly the same manner. To confirm this, it is sufficient to note that if the function $V(t, y)$ admits an infinitely small upper bound, then the y-fixed-sign function $U(t, x)$ is a fixed-sign function in the region $V(t, y) > 0$. The function $V^{\cdot}(t, x) = \lambda V(t, y) + W(t, x)$ is the y-fixed-sign function in the region where $V(t, y)$ assumes the sign coinciding with the sign of the constant-sign function $W(t, x) \not\equiv 0$. If on the other hand $W \equiv 0$, then $V^{\cdot}$ is fixed-sign in both regions $V > 0$ and $V < 0$ [6].

BIBLIOGRAPHY

1. Liapunov, A. M., Coll. Works, Vol. 2, (p. 272), M.-L., Izd. Akad. Nauk SSSR, 1956.
2. Krasovskii, N. N., Stability of Motion, Stanford Univ. Press, Stanford, Cal., 1963.
3. Rumiantsev, V. V., Stability of motion with respect to a part of the variables. Vest. MGU, Ser. matem., mekhan., fiz., khim., astron., №4, 1957.
4. Corduneanu, C., Sur la stabilité partielle. Rev. Roumaine math. pures appl. Vol. 9, №3, 1964.
5. Peiffer, K. and Rouche, N., Liapunov's second method applied to partial stability. J. Mec. Vol. 8, №2, 1969.
6. Chetaev, N. G., Stability of Motion, 2nd ed. M., Gostekhizdat, 1955.
7. Yoshizawa, T., Stability theory Liapunov's second method. Tokyo, Math. Soc. Japan, p. 223, 1966.
8. Rumiantsev, V. V., On stability of motion of nonholonomic systems. PMM Vol. 31, №2, 1967.
9. Risito, C., Sulla stabilita asintotica parziale. Annali di Matematica pura ed applicatu. Ser. 4, Vol. 84, 279 - 292, 1970.

Translated by L. K.